FUNCTIONS WEAKER THAN CONTINUOUS FUNCTIONS

by

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The author studied the compositions, restrictions and extensions of several classes of functions weaker than continuous functions. With appropriate limitations imposed on these functions, theorems which resemble theorems of continuous functions were proved.

APPROVAL PAGE

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INTRODUCTION

Many properties which are characteristic of continuous function remain valid for several classes of non-continuous functions if appropriate limitations are imposed. Some problems of topology may be solved only through the study of non-continuous functions. In particular, Almost-Continuous Mappings by Singal and Singal [2] turn out to be the natural tool for studying almost-compact spaces (A space is said to be almost-compact if each open cover has a finite subfamily whose closures cover the space) of Alexandroff and Urysohn and also nearly-compact spaces (A space is said to be nearly-compact if every open cover has a finite subfamily the interiors of the closures of whose members cover the space) in as much as every almost-continuous image of an almost-compact space is almost-compact and every almost-continuous open image of a nearly-compact space is nearly-compact [3].

One of the aims of this thesis is to define some functions weaker than continuous functions in a topological space and to study the compositions, restrictions and extensions of these functions which are closely parallel to the elementary properties of continuous functions. The ideas behind these results originated in [1], [2] and [6].

In Chapter I, Almost Continuous Mapping is defined and theorems of compositions and restrictions are proved.

In Chapter II, Almost-Continuous Mapping, weakly-continuous functions and θ-continuous functions are defined. The basic theorem on Almost-Continuous Mapping is proved.

In Chapter III, Somewhat Continuous Function is defined, and the basic theorem on Somewhat Continuous Function is proved.

CHAPTER I

STALLINGS' ALMOST CONTINUOUS MAPPING

DEFINITION 1.1. If $f : X \to Y$ is a function of topological spaces, then that f is almost continuous means that for any open set $N \subset X \times Y$, if $\Gamma(f) \subset N$, [$\Gamma(f)$ denotes the graph of f] there exists a continuous function $g : X \to Y$ such that $\Gamma(g) \subset N$ [1].

EXAMPLE 1.1. Let R be the set of real numbers with the usual topology. Let $f : R \to R$ be defined as follows:

$$f(x) = \sin \frac{1}{x} \text{ if } x \neq 0,$$

$$f(0) = 0.$$

Then f is almost continuous, but not continuous.

PROOF: Let N be an open subset of $R \times R$ which contains the graph of f. Then the point $(0, f(0)) = (0, 0)$ is an element of N. Hence there exists an open disc $S(0, r)$ with radius r and center 0, such that, $S(0, r) \subset N$. Let n be a positive integer such that $\frac{1}{n\pi} < r$. Define $g : R \to R$ by

$$g(x) = f(x) \text{ if } |x| \geq \frac{1}{n\pi},$$

$$g(x) = 0 \text{ if } |x| < \frac{1}{n\pi}.$$

Then g is continuous on R. Also, $\Gamma(g) \subset N$. For if $|x| \geq \frac{1}{n\pi}$, then $(x, g(x)) = (x, f(x)) \in N$ by assumption and if $|x| < \frac{1}{n\pi}$, then $(x, g(x)) = (0, 0) \varepsilon N$. Hence f is almost continuous. By definition f is not continuous.

It is clear that every continuous function is almost continuous.

THEOREM 1.1. If $f : X \to Y$ is almost continuous and $g : Y \to Z$ is continuous where X, Y and Z are topological spaces, then $gof : X \to Z$ is almost continuous.

PROOF: Let N be an open set of $X \times Z$, such that $\Gamma(gof) \subset N$. Let $g_* : X \times Y \to X \times Z$ be defined as $g_* (x, y) = (x, g(y))$. Then $g_*^{-1} (N)$ is an open set of $X \times Y$ and $\Gamma(f) \subset g_*^{-1} (N)$. Since f is almost continuous there is a continuous $F : X \to Y$ such that $\Gamma(F) \subset g_*^{-1} (N)$. Hence $goF : X \to Z$ is continuous and $\Gamma(goF) \subset N$.

THEOREM 1.2. If $f : X \to Y$ is almost continuous and C is a closed subset of X, then $f|_C : C \to Y$ is almost continuous.

PROOF: Let N be an open subset of $C \times Y$ such that $\Gamma(f|_C) \subset N$. Then there is an open set N' in $X \times Y$ such that $N = N' \cap (C \times Y)$. The set $N' \cup [(X - C) \times Y]$ is then open in $X \times Y$ and $\Gamma(f) \subset N' \cup [(X - C) \times Y]$. Hence there is a continuous $F : X \to Y$, $\Gamma(F) \subset N' \cup [(X - C) \times Y]$. Clearly, $F|_C : C \to Y$ is continuous and $\Gamma(F|_C) \subset N$.

THEOREM 1.3. Let X be a compact Hausdorff space, Y a Hausdorff space and Z a topological space. If $f : X \to Y$ is continuous and $g : Y \to Z$ is almost continuous, then $gof : X \to Z$ is almost continuous.

PROOF: First note that if Y is replaced by $f(X)$, then $f : X \to f(X)$ is continuous. $f(X)$ is the continuous image of the compact set X and thus is compact. Since $f(X)$ is a compact subset of the Hausdorff space Y, $f(X)$ is closed in Y.
So $g|_{f(X)} : f(X) \to Z$ is almost continuous by Theorem 1.2. Thus we can assume that f maps X onto Y. Let N be an open set of $X \times Z$ such that $\Gamma(gof) \subset N$; let $f_* : X \times Z \to Y \times Z$ be defined by $f_* (x, z) = (f(x), z)$. Now, by definition of graph and gof, $\Gamma(gof) = \{(x, g(f(x))) \mid x \in X\}$ and $\Gamma(g) = \{(f(x), g(f(x))) \mid x \in X\}$ since f is onto. So $f_* (x, g(f(x))) = (f(x), g(f(x)))$ by definition of f_*. Thus $f_* (\Gamma(gof)) \subset \Gamma(g)$. Let $(f(x), g(f(x)))$ be an arbitrary point in $\Gamma(g)$. Then $f_* (x, g(f(x))) = (f(x), g(f(x)))$. Hence $(f(x), g(f(x))) \in f_* (\Gamma(gof))$. Therefore $\Gamma(g) \subset f_* (\Gamma(gof))$. Thus $f_* (\Gamma(gof)) = \Gamma(g)$. Now, for any $y \in Y$, $f^{-1}(y)$ is a compact subset of X; for any $x \in f^{-1}(y)$ let N_x be an open set of X containing x and M_x be an open set in Z containing $gof(x) = g(y)$, such that $N_x \times M_x \subset N$; a finite number, say $N_1, \cdots, N_k$ of these N_x cover $f^{-1}(y)$; let $M_1, \cdots, M_k$ be the corresponding M_x. Then let $U_y = Y - f\,(X - \bigcup_{i=1}^{k} N_i)$; and let $W_y = U_y \times [\bigcap_{i=1}^{k} M_i]$. Then U_y is an open subset of Y containing y; hence W_y is an open subset of $Y \times Z$ containing $(y, g(y))$. Let $(x, z) \in f_*^{-1} (W_y)$. Then $f_*(x, z) = (f(x), z) \in W_y = U_y \times [\bigcap_{i=1}^{k} M_i]$. So $f(x) \in U_y$ and $z \in M_i$ where $i = 1,\ldots,k$.

$f(x) \notin f(X - \bigcup_{i=1}^{k} N_i)$ since $U_y = Y - f(X - \bigcup_{i=1}^{k} N_i)$. So $x \notin X - \bigcup_{i=1}^{k} N_i$. Hence $x \in \bigcup_{i=1}^{k} N_i$ which implies $x \in N_{i_o}$. Also, $z \in M_{i_o}$ since $z \in M_i$ where $i = 1,\ldots,k$. Thus $(x, z) \in N_{i_o} \times M_{i_o} \subset N$. Therefore $f_*^{-1}(W_y) \subset N$. Let $W = \bigcup_{y \in Y} W_y$. Then W is an open subset of $Y \times Z$, and $\Gamma(g) \subset W$. Therefore there is a continuous $G : Y \to Z$ such that $\Gamma(G) \subset W$. Then $Gof : X \to Z$ is continuous. $f_*(\Gamma(Gof)) = f_*(\{(x, z)\}) = \{(f(x), z)\} = \{(y, z)\} = \Gamma(G) \subset W$. Thus $f_*(\Gamma(Gof)) \subset W$. Hence $\Gamma(Gof) \subset f_*^{-1}(f_*(\Gamma(Gof))) \subset f^{-1}(W) \subset N$. Therefore gof is almost continuous.

CHAPTER II

SINGAL AND SINGAL'S ALMOST-CONTINUOUS MAPPINGS

DEFINITION 2.1. A mapping $f : X \to Y$ is said to be almost-continuous at a point $x \in X$, if for every neighborhood M of $f(x)$ there is a neighborhood N of x such that $f(N) \subset \mathrm{Int}\,(\overline{M})$ [2].

It is clear that if $f : X \to Y$ is continuous at a point $x \in X$, then it is almost-continuous at x. But the converse of this statement may not be true, as the following example shows.

EXAMPLE 2.1. Let R be the set of real numbers and let T consists of $\emptyset$, R and the complements of all countable subsets of R. Let $X = \{a, b\}$ and let $T^* = \{X, \emptyset, \{a\}\}$. Let $f : (R, T) \to (X, T^*)$ be defined as follows:

$$f(x) = \begin{cases} a, & \text{if } x \text{ is rational,} \\ b, & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is almost continuous but not continuous.

PROOF: Let $x \in R$ be rational. Then X and $\{a\}$ are the only two neighborhoods of $f(x) = a$. If $M = X$, then $\overline{M} = X$. Thus $\mathrm{Int}\,(\overline{M}) = X$. Therefore $f(N) \subset \mathrm{Int}\,(\overline{M})$ for any neighborhood N of x. If $M = \{a\}$, then $\overline{\{a\}} = X$. Thus $\mathrm{Int}\,(\overline{\{a\}}) = X$. Hence $f(N) \subset \mathrm{Int}\,(\overline{\{a\}})$ for any neighborhood N of x. Now, let $x \in R$ be irrational. Then $f(x) = b$, and X is the only neighborhood of

$f(x)$. Therefore f is almost-continuous at each point of R, but f is not continuous at $x \in R$ if x is rational.

DEFINITION 2.2. A mapping $f : X \to Y$ is said to be almost-continuous if it is almost-continuous at each point x of X.

An almost-continuous mapping may fail to be continuous. The mapping f of example 2.1 is an almost-continuous mapping which is not continuous. The following is another example of such a mapping.

EXAMPLE 2.2. Let R be the set of real numbers and let T consist of $\emptyset$, R and the complements of all countable subsets of R. Let U denote the usual topology for R. Let i be the identity mapping of (R, U) onto (R, T). Then $i : (R, U) \xrightarrow{\text{onto}} (R, T)$ is almost-continuous but not continuous.

PROOF: Let $x \in R$ and M be any neighborhood of x in (R, T). Then M is an uncountable set, and hence $\overline{M} = R$ since the only closed sets in (R, T) are all countable sets of R. Thus $i\,(N) \subset \overline{M} = R$ for any neighborhood N of x in (R, U).

Now, i is not continuous at any $x \in R$. For if x is rational, then $I \cup \{x\}$ is an open set in T containing x. [I = irrationals]. But $i^{-1}\,(I \cup \{x\}) = I \cup \{x\}$, which is not open in U. If x is irrational, then I is an open set in T containing x, and $i^{-1}\,(I) = I$ is not open in U.

Before stating and proving one of the main theorems, we need the following preliminaries.

DEFINITION 2.3. A set A is called regularly-open if $A = \mathrm{Int}\,(\overline{A})$; A set U is called regularly-closed if $U = \overline{\mathrm{Int}\,(U)}$.

<u>DEFINITION 2.4</u>. Let D be a nonempty set. The relation $\geq$ directs D iff the following three conditions hold:

(a) For every $a \in D$, $a \geq a$.

(b) If $a \geq b$ and $b \geq c$, then $a \geq c$.

(c) For each $a, b \in D$, there exists an element $c \in D$ such that $c \geq a$ and $c \geq b$.

A <u>directed set</u> is a set D together with a relation $\geq$ that directs D and is denoted by $(D, \geq)$.

<u>DEFINITION 2.5</u>. Let X be any non-empty set, and let D be a directed set. Let $f : D \to X$ be a function. Then identifying f with its range we have $f = \{x_\lambda\}_{\lambda \in D}$ where $x_\lambda = f(\lambda)$, $\lambda \in D$. $\{x_\lambda\}_{\lambda \in D}$ is called a <u>net</u> in X.

<u>DEFINITION 2.6</u>. Let (X, T) be a topological space. Let $\{x_\lambda\}_{\lambda \in D}$ be a net in X, and let $x \in X$. We say that $\{x_\lambda\}_{\lambda \in D}$ <u>converges to x</u> if given an open set U containing x, $\{x_\lambda\}_{\lambda \in D}$ is eventually in U, that is, there exists $\lambda \in D$ such that $\mu \geq \lambda$ implies $x_\mu \in U$.

<u>LEMMA 2.1</u>. A is a regularly-open subset of Y iff $Y - A$ is regularly-closed.

PROOF: Let A be a regularly-open subset of Y. Then $Y - A = Y - \text{Int}\,(\overline{A})$. $\text{Int}\,(\overline{A}) = Y - (\overline{Y - \overline{A}})$, so $\overline{Y - \overline{A}} = Y - \text{Int}\,(\overline{A})$. Hence $Y - A = \overline{Y - \overline{A}}$. $\text{Int}\,(Y - A) = Y - \overline{A}$, so $\overline{A} = Y - \text{Int}\,(Y - A)$. Hence $\overline{Y - \overline{A}} = \overline{Y - (Y - \text{Int}\,(Y - A))} = \overline{\text{Int}\,(Y - A)}$. So $Y - A = \overline{\text{Int}\,(Y - A)}$. Therefore $Y - A$ is regularly-closed.

Now, suppose $Y - A$ is regularly-closed. Then $A = Y - (Y - A) = Y - \overline{\text{Int } (Y - A)}$. $\text{Int } (Y - A) = Y - \overline{A}$, so $\overline{\text{Int } (Y - A)} = \overline{Y - \overline{A}}$. Thus $Y - \overline{\text{Int } (Y - A)} = Y - \overline{(Y - \overline{A})} = \text{Int } (\overline{A})$. Hence $A = \text{Int } (\overline{A})$. Therefore A is regularly-open.

<u>LEMMA 2.2</u>. If A is closed, then $\text{Int } (A)$ is a regularly-open set.

PROOF: Let $V = \text{Int } (A)$. Then $\overline{V} \subset A$, and hence $\text{Int } (\overline{V}) \subset \text{Int } (A) = V$. Also $V \subset \overline{V}$ implies $V \subset \text{Int } (\overline{V})$ and hence $V = \text{Int } (\overline{V})$ so V is regularly-open. Since $V = \text{Int } (A)$, therefore $\text{Int } (A)$ is regularly-open.

<u>THEOREM 2.1</u>. For a mapping $f : X \to Y$, the following are equivalent:

(a) f is almost-continuous.

(b) Inverse image of every regularly-open subset of Y is an open subset of X.

(c) Inverse image of every regularly-closed subset of Y is a closed subset of X.

(d) For each point x of X and for each regularly-open neighborhood M of $f(x)$, there is a neighborhood N of x such that $f(N) \subset M$.

(e) $f^{-1} (A) \subset \text{Int } [f^{-1} (\text{Int } (\overline{A}))]$ for every open subset A of Y.

(f) $\overline{[f^{-1} (\overline{\text{Int } (B)})]} \subset f^{-1} (B)$ for every closed subset B of Y.

(g) For any point $x \varepsilon X$ and for any net $\{x_\lambda\}_{\lambda \varepsilon D}$ which converges to x, the net $\{f(x_\lambda)\}_{\lambda \varepsilon D}$ is eventually in each regularly-open set containing $f(x)$.

PROOF: (a) $\Longrightarrow$ (b). Let U be any regularly-open subset of Y and let $x \varepsilon f^{-1}(U)$. Then $f(x) \varepsilon U$. Therefore there exists an open set V in X such that $x \varepsilon V$ and $f(V) \subset \text{Int}\,(\overline{U}) = U$. Thus, $x \varepsilon V \subset f^{-1}(U)$ and hence $f^{-1}(U)$ is the union of open sets which is open.

(b) $\Longrightarrow$ (c). Let A be any regularly-closed subset of Y. By Lemma 2.1 $Y - A$ is regularly-open and therefore $f^{-1}(Y - A)$ is open, that is, $X - f^{-1}(A)$ is open. Hence $f^{-1}(A)$ is closed.

(c) $\Longrightarrow$ (d). Since M is regularly-open, $Y - M$ is regularly-closed by Lemma 2.1. Consequently $f^{-1}(Y - M)$ is closed, that is, $f^{-1}(M)$ is open. Also $x \varepsilon f^{-1}(M) = N$ (say). Then N is a neighborhood of x such that $f(N) \subset M$.

(d) $\Longrightarrow$ (e). Let A be an open subset of Y. Then $\overline{A}$ is a closed set. Let $x \varepsilon f^{-1}(A)$. Then by Lemma 2.2, $\text{Int}\,(\overline{A})$ is a regularly-open neighborhood of $f(x)$, since A is open. Then, there exists an open neighborhood N of x such that $f(N) \subset \text{Int}\,(\overline{A})$. Thus $x \varepsilon N \subset f^{-1}(\text{Int}\,(\overline{A}))$. This means that $x \varepsilon \text{Int}\,[f^{-1}(\text{Int}\,(\overline{A}))]$. Hence $f^{-1}(A) \subset \text{Int}\,[f^{-1}(\text{Int}\,(\overline{A}))]$.

(e) $\Longrightarrow$ (f). Let B be a closed subset of Y. Then $Y - B$ is open. Since $Y - B$ is open, therefore
$f^{-1}(Y - B) \subset \text{Int}\,[f^{-1}(\text{Int}\,(\overline{Y - B}))]$ by hypothesis. Therefore
$X - f^{-1}(B) \subset \text{Int}\,[f^{-1}(\text{Int}\,(\overline{Y - B}))]$ since $f^{-1}(Y - B) = X - f^{-1}(B)$.

Thus $X - \text{Int}\,[f^{-1}\,(\text{Int}\,(\overline{Y - B}))] \subset f^{-1}\,(B)$.

But $\text{Int}\,[f^{-1}\,(\text{Int}\,(\overline{Y - B}))] \subset f^{-1}\,(\text{Int}\,(\overline{Y - B}))$, and therefore

$\overline{X - f^{-1}\,(\text{Int}}\,(\overline{Y - B})) \subset X - \text{Int}\,[f^{-1}\,(\text{Int}\,(\overline{Y - B}))]$

since $X - \text{Int}\,[f^{-1}\,(\text{Int}\,(\overline{Y - B}))]$ is closed.

Hence $\overline{X - f^{-1}\,(\text{Int}\,(\overline{Y - B}))} \subset f^{-1}\,(B)$.

$X - f^{-1}\,(\text{Int}\,(\overline{Y - B}) = f^{-1}\,(Y - \text{Int}\,(\overline{Y - B}))$. But $Y - \text{Int}\,(\overline{Y - B})$ $= \overline{\text{Int}\,(B)}$ since $Y - \text{Int}\,(\overline{Y - B}) = \overline{Y - (\overline{Y - B})} = \overline{\text{Int}\,(B)}$.

Therefore $\overline{X - f^{-1}\,(\text{Int}\,(\overline{Y - B}))} = \overline{f^{-1}\,(\overline{\text{Int}\,(B)})}$. Hence $\overline{f^{-1}\,(\overline{\text{Int}\,(B)})}$ $\subset f^{-1}\,(B)$.

(f) $\Longrightarrow$ (g). Let N be any regularly-open set containing $f(x)$. Then $Y - N$ being closed, $\overline{[f^{-1}\,\text{Int}\,(\overline{Y - N}))]} \subset f^{-1}\,(Y - N)$. Since $Y - N$ is regularly-closed by Lemma 2.1, therefore $\overline{[f^{-1}\,(Y - N)} \subset X - f^{-1}\,(N)$. This means that $f^{-1}\,(N) \subset \text{Int}\,[f^{-1}\,(N)]$. Thus $f^{-1}\,(N)$ is an open set containing x. Since the net $\{x_\lambda\}_{\lambda \in D}$ converges to x, therefore there exists $\lambda_0 \in D$ such that for all $\lambda \geq \lambda_0$ (D is directed by '$\geq$') $x_\lambda \in f^{-1}\,(N)$. This means that $f(x_\lambda) \in N$ for all $\lambda \geq \lambda_0$, that is, the net $\{f(x_\lambda)\}_{\lambda \in D}$ is eventually in N.

(g) $\Longrightarrow$ (a). Suppose that f is not almost-continuous. Then, there is an open set V containing $f(x)$ such that for any open set U containing x, $f(U) \cap [Y - (\text{Int}\,(\overline{V}))] \neq \emptyset$. This implies that $U \cap f^{-1}\,[Y - (\text{Int}\,(\overline{V}))] \neq \emptyset$ for any open set U containing x. The family $\mathcal{U}$ of all open sets U containing x is directed by set inclusion. For any $U \in \mathcal{U}$ choose a point x_U belonging to

$U \cap f^{-1}$ [Int $\overline{(Y - V)}$]. Then $\{x_U\}_{U \in \mathcal{U}}$ is a net in X which converges to x and is such that no $f(x_U)$ is in Int $(\overline{V})$. Thus $\{f(x_U)\}_{U \in \mathcal{U}}$ is not eventually in the regularly-open set Int $(\overline{V})$, which contradicts the hypothesis.

THEOREM 2.2. If f is an open continuous mapping of X onto Y and if g is a mapping of Y into Z, then gof is almost-continuous iff g is almost-continuous.

PROOF: First, let gof be almost-continuous. Let A be a regularly-open subset of Z. Since gof is almost-continuous, therefore $(gof)^{-1}$ (A) is open, that is, f^{-1} $(g^{-1}$ (A)) is open by Theorem 2.1. Also f is open. Therefore $f[f^{-1}$ $(g^{-1}$ (A))] is open, that is, g^{-1} (A) is open and consequently g is almost-continuous.

Now, let g be almost-continuous and let A be any regularly-open subset of Z. Then g^{-1} (A) is an open subset of Y. Since f is continuous, therefore f^{-1} $(g^{-1}$ (A)) is an open subset of X, that is, $(gof)^{-1}$ (A) is an open subset of X. Hence gof is almost-continuous.

LEMMA 2.3. Let $f : (X, T) \to (Y, T^*)$ be a homeomorphism. If W is regularly-open in X, then f(W) is regularly-open in Y.

PROOF: f(W) is open since f is an open map. Hence $f(W) \subset$ Int $(\overline{f(W)})$. Let $y \in$ Int $(\overline{f(W)})$. Then there exists an open set U in Y such that $y \in U \subset \overline{f(W)}$. But $\overline{f(W)} = f(\overline{W})$ since f is homeomorphism. Therefore, $y \in U \subset f(\overline{W})$ and hence $f^{-1}(U) \subset \overline{W}$. Thus f^{-1} (U) $\subset$ Int $(\overline{W})$ since f^{-1} (U) is open. By hypothesis,

$\text{Int}\ (\overline{W}) = W$. So $f^{-1}\ (U) \subset W$ and hence $U \subset f(W)$. Therefore $y \in f(W)$. Hence we can conclude that $\text{Int}\ (\overline{f(W)}) \subset f(W)$, and consequently $\text{Int}\ (\overline{f(W)}) = f(W)$. Therefore $f(W)$ is regularly-open.

THEOREM 2.3. Let $f : X \to Y$ be almost-continuous and $g : Y \to Z$ be a homeomorphism. Then $gof : X \to Z$ is almost-continuous.

PROOF: Let V be a regularly-open subset of Z. By Lemma 2.3, it follows that $g^{-1}\ (V)$ is a regularly-open subset of Y. Since f is almost-continuous, $f^{-1}\ (g^{-1}\ (V))$ is an open subset of X. But $f^{-1}\ (g^{-1}\ (V)) = (gof)^{-1}\ (V)$. Hence gof is almost-continuous.

REMARK 2.1. The composition of two almost-continuous functions may fail to be almost-continuous as the following example shows.

EXAMPLE 2.3. Let $X = \{a, b, c, d\}$, $S = \{\emptyset, X, \{a, d\}, \{c\}, \{a, c, d\}\}$ and $T = \{\emptyset, X, \{a, b, c\}\}$. Let $f : (X, S) \to (X, T)$ be defined as follows:

$$f(x) = x,$$

and $g : (X, T) \to (X, S)$ be defined as follows:

$$g(a) = c$$

$$g(b) = c$$

$$g(c) = c$$

$$g(d) = b.$$

The regularly-open sets of T are $\emptyset$ and X, and $f^{-1}\ (\emptyset) = \emptyset$ and $f^{-1}\ (X) = X$ which are open in S. The regularly-open sets of S are $\{a, d\}$, $\{c\}$, X, and $\emptyset$. $g^{-1}\ (\{c\}) = \{a, b, c\}$ and $g^{-1}\ (\{a, d\}) = \emptyset$

which are open in T. So by Theorem 2.1 (a) and (b) f and g are almost-continuous. Now, look at $gof : (X, S) \to (X, S)$. $(gof)^{-1}(\{c\}) = f^{-1}(g^{-1}(\{c\})) = f^{-1}(\{a, b, c\}) = \{a, b, c\}$ which is not open in S. Therefore gof is not almost-continuous.

THEOREM 2.4. Every restriction of an almost-continuous mapping is almost-continuous.

PROOF: Let f be an almost-continuous mapping of X into Y and let A be any subset of X. For any regularly-open subset S of Y, $(f|_A)^{-1}(S) = A \cap f^{-1}(S)$. But, f being almost-continuous, $f^{-1}(S)$ is open and hence $A \cap f^{-1}(S)$ is a relatively open subset of A, that is, $(f|_A)^{-1}(S)$ is an open subset of A. Hence $f|_A$ is almost-continuous.

THEOREM 2.5. Let f map X into Y and let x be a point of X. If there exists an open set N containing x such that the restriction of f to N is almost-continuous at x, then f is almost-continuous at x.

PROOF: Let U be any regularly-open set containing $f(x)$. Since $f|_N$ is almost-continuous at x, therefore, there is an open set V_1 such that $x \in N \cap V_1$ and $f(N \cap V_1) \subset U$. Since $N \cap V_1$ is an open set containing x, we can conclude that f is almost-continuous at x by Theorem 2.1 (a) and (d).

THEOREM 2.6. If f is a mapping of X into Y and $X = X_1 \cup X_2$, where X_1 and X_2 are closed and $f|_{X_1}$ and $f|_{X_2}$ are almost-continuous, then f is almost-continuous.

PROOF: Let A be a regularly-closed subset of Y. Then, since $f|_{X_1}$ and $f|_{X_2}$ are both almost-continuous, therefore $(f|_{X_1})^{-1}(A)$ and $(f|_{X_2})^{-1}(A)$ are both closed in X_1 and X_2 respectively. Since X_1 and X_2 are closed subsets of X, therefore $(f|_{X_1})^{-1}(A)$ and $(f|_{X_2})^{-1}(A)$ are also closed subsets of X. Also $f^{-1}(A)$ $= (f|_{X_1})^{-1}(A) \cup (f|_{X_2})^{-1}(A)$. Thus $f^{-1}(A)$ is the union of two closed subsets and is therefore closed. Hence by Theorem 2.1 (a) and (c) f is almost-continuous.

<u>THEOREM 2.7</u>. If f is a mapping of X into Y and $X = X_1 \cup X_2$ and if $f|_{X_1}$ and $f|_{X_2}$ are both almost-continuous at a point x belonging to $X_1 \cap X_2$, then f is almost-continuous at x.

PROOF: Let U be any regularly-open set containing $f(x)$. Since $x \in X_1 \cap X_2$ and $f|_{X_1}$, $f|_{X_2}$ are both almost-continuous at x, therefore there exist open sets V_1 and V_2 such that $x \in X_1 \cap V_1$ and $f(X_1 \cap V_1) \subset U$, and $x \in X_2 \cap V_2$ and $f(X_2 \cap V_2) \subset U$. Now, since $X = X_1 \cup X_2$, therefore $f(V_1 \cap V_2) = f(X_1 \cap V_1 \cap V_2)$ $\cup\ f(X_2 \cap V_1 \cap V_2) \subset f(X_1 \cap V_1) \cup f(X_2 \cap V_2) \subset U$. Thus $V_1 \cap V_2 = V$ is an open set containing x such that $f(V) \subset U$ and hence f is almost continuous at x.

<u>DEFINITION 2.7</u>. A mapping $f : X \to Y$ is said to be <u>weakly-continuous</u> if for each point $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset \overline{V}$ [4].

REMARK 2.2. Obviously, every almost-continuous mapping is weakly-continuous. But a weakly-continuous mapping may fail to be almost-continuous. The following is an example.

EXAMPLE 2.4. Let R be the set of real numbers and let T consist of $\emptyset$, R and the complements of all countable subsets of R. Let $X = \{a, b, c\}$ and let $T^* = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Let $f : (R, T) \to (X, T^*)$ be defined as follows:

$$f(x) = \begin{cases} a, & \text{if } x \text{ is rational,} \\ b, & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is a weakly-continuous mapping which is not almost-continuous at any rational point.

PROOF: First, let $x \in R$ be rational. Then $f(x) = a$. Now X, $\{a\}$, $\{a, c\}$ are the only open neighborhoods of $f(x) = a$. If $V = X$, then clearly there exists a neighborhood U of x such that $f(U) \subset \overline{V}$, that is, take $U = R$. If $V = \{a\}$, then $\overline{V} = \{a, b\}$. Thus we may take R as our neighborhood of x since $f(R) = \{a, b\} = \overline{V}$. Finally if $V = \{a, c\}$ then $\overline{V} = X$. Again take $U = R$. Then clearly $f(U) \subset \overline{V}$. Therefore f is weakly-continuous at x.

Now suppose x is irrational. Then $f(x) = b$ and X is the only neighborhood of b. Thus, if $U = R$, $V = X$, then $f(U) \subset \overline{V} = V$. So f is weakly-continuous at x. Therefore f is weakly-continuous.

To see that f is not almost-continuous at x if x is rational, note $V = \{a\}$ is a neighborhood of $f(x) = a$ and Int $(\overline{V}) = V$, and that every neighborhood U of x contain both rational and irrational numbers and consequently, $f(U) = \{a, b\} \not\subset \text{Int } (\overline{V}) = V$. Therefore f is a weakly-continuous mapping which is not almost-continuous at any rational point.

However, we have the following:

<u>THEOREM 2.8</u>. If $f : X \to Y$ is a weakly-continuous open mapping, then f is almost-continuous.

PROOF: Let $x \in X$ and let M be any neighborhood of $f(x)$. Since f is weakly-continuous, there is an open neighborhood N of x such that $f(N) \subset \overline{M}$. Since f is open, therefore $f(N)$ is open. Then $f(N) \subset \text{Int } (\overline{M})$ and consequently f is almost-continuous.

<u>COROLLARY 2.1</u>. An open mapping is almost-continuous iff it is weakly-continuous.

<u>DEFINITION 2.8</u>. A mapping $f : X \to Y$ is said to be <u>θ-continuous</u> if for each point $x \in X$ and each neighborhood U of $f(x)$, there is a neighborhood V of x such that $f(\overline{V}) \subset \overline{U}$ [5].

<u>REMARK 2.3</u>. It is clear that every θ-continuous mapping is weakly-continuous. A θ-continuous mapping may fail to be almost-continuous. In fact, the mapping f defined in example 2.4 is θ-continuous but not almost continuous. We do not know, however whether every almost-continuous mapping is θ-continuous or not.

CHAPTER III

SOMEWHAT CONTINUOUS FUNCTIONS

DEFINITION 3.1. Let (X, S) and (Y, T) be topological spaces. A function $f : (X, S) \to (Y, T)$ is said to be somewhat continuous provided that if $U \in T$ and $f^{-1}(U) \neq \emptyset$, then there is a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$ [6].

REMARK 3.1. It is clear that every continuous function is somewhat continuous, but a somewhat continuous function need not be continuous as the following example shows.

EXAMPLE 3.1. Let $X = \{a, b, c, d\}$, let $S = \{\emptyset, X, \{a, d\}, \{c\}, \{a, c, d\}\}$, and let $T = \{\emptyset, X, \{a, b, c\}\}$. Then the identity function $i : (X, S) \to (X, T)$ is somewhat continuous but not continuous.

PROOF: The open sets of T are $\emptyset$, X and $\{a, b, c\}$. $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X \subset X \in S$, and $f^{-1}(\{a, b, c\}) = \{a, b, c\}$ and $\{c\} \subset \{a, b, c\}$. Hence i is somewhat continuous. To see that i is not continuous, note $f^{-1}(\{a, b, c\}) = \{a, b, c\} \notin S$.

THEOREM 3.1. If $f : (X, S) \to (Y, T)$ and $g : (Y, T) \to (Z, U)$ are somewhat continuous functions and f is onto, then $gof : (X, S) \to (Z, U)$ is somewhat continuous.

PROOF: Let $W \in U$ such that $(gof)^{-1}(W) \neq \emptyset$. Since $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$, $g^{-1}(W) \neq \emptyset$. Thus there is an $A \in T$ such that $A \neq \emptyset$ and $A \subset g^{-1}(W)$. Since f is onto, $f^{-1}(A) \neq \emptyset$ and therefore there is a $B \in S$ such that $B \neq \emptyset$ and $B \subset f^{-1}(A)$.

Now, $B \subset f^{-1}(A) \subset f^{-1}(g^{-1}(W)) = (gof)^{-1}(W)$, and hence gof is somewhat continuous.

THEOREM 3.2. If $f : (X, S) \to (Y, T)$ is somewhat continuous and $g : (Y, T) \to (Z, U)$ is continuous, then $gof : (X, S) \to (Z, U)$ is somewhat continuous.

PROOF: Let $W \in U$ such that $(gof)^{-1}(W) \neq \emptyset$. Since g is continuous $g^{-1}(W) \in T$. Since $(gof)^{-1}(W) = f^{-1}(g^{-1}(W)) \neq \emptyset$, there is an $A \in S$ such that $A \neq \emptyset$ and $A \subset f^{-1}(g^{-1}(W)) = (gof)^{-1}(W)$. Hence gof is somewhat continuous.

REMARK 3.2. If $f : (X, S) \to (Y, T)$ is continuous and $g : (Y, T) \to (Z, U)$ is somewhat continuous, then $gof : (X, S) \to (Z, U)$ may not be somewhat continuous as the following example shows.

EXAMPLE 3.2. Let $X = \{a, b, c, d\}$, $S = \{\emptyset, X\}$, $T = \{\emptyset, X, \{b\}\}$, and $U = \{\emptyset, X, \{b, c, d\}\}$. Define $f : (X, S) \to (X, T)$ as follows:

$$f(a) = a$$
$$f(b) = a$$
$$f(c) = c$$
$$f(d) = d\ .$$

Define $g : (X, T) \to (X, U)$ by $g(x) = x$ for all $x \in X$. Then $gof : (X, S) \to (X, U)$ is not somewhat continuous.

PROOF: f is continuous since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{b\}) = \emptyset$ which belong to S. g is somewhat continuous since

$g^{-1}(X) = X$ and $X \subset X$, and $g^{-1}(\{b, c, d\}) = \{b, c, d\}$ and $\{b\} \subset \{b, c, d\}$. Now, look at $(gof)^{-1}(\{b, c, d\}) = f^{-1}(g^{-1}(\{b, c, d\})) = f^{-1}(\{b, c, d\}) = \{c, d\}$. There is no nonempty set belonging to S that is a subset of $\{c, d\}$. Therefore gof is not somewhat continuous.

<u>THEOREM 3.3</u>. If $f : (X, S) \rightarrow (Y, T)$ is a function, then the following are equivalent:

(a) f is somewhat continuous.

(b) If C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper closed subset D of X such that $D \supset f^{-1}(C)$.

(c) If M is a dense subset of X, then $f(M)$ is a dense subset of $f(X)$.

PROOF: (a) $\Rightarrow$ (b). Suppose f is somewhat continuous. Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y - C$ is open and $f^{-1}(Y - C) \neq \emptyset$. Since f is somewhat continuous, there is a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(Y - C)$. Now $X - V$ is closed and since $V \neq \emptyset$, $X - V \neq X$. Let $x \in f^{-1}(C)$. Then $f(x) \in C$ and thus $f(x) \notin Y - C$. Therefore $x \notin f^{-1}(Y - C)$ and thus $x \notin V$. Therefore $x \in X - V$. Hence $X - V \supset f^{-1}(C)$.

(b) $\Rightarrow$ (c). Suppose f has property (b). Let M be a dense subset of X. Let U be a nonempty open subset of $f(X)$ under the relative topology. Then there is an element $V \in T$ such that

$U = V \cap f(X)$. Therefore $Y - V$ is closed and since $f^{-1}(V) \neq \emptyset$, $f^{-1}(Y - V) \neq X$. Thus there is a proper closed subset D of X such that $D \supset f^{-1}(Y - V)$. Since $X - D$ is a nonempty open set and M is a dense subset of X, $X - D$ contains a point p of M. Since $p \notin D$, $p \notin f^{-1}(Y - V)$ and thus $f(p) \notin Y - V$ and $f(p) \varepsilon V$. Therefore $f(p) \varepsilon V \cap f(X) = U$. Hence U contains the point $f(p)$ of $f(M)$ and thus $f(M)$ is dense in $f(X)$.

(c) $\Longrightarrow$ (a). Suppose f has property (c). Let $U \varepsilon T$ such that $f^{-1}(U) \neq \emptyset$. Suppose $f^{-1}(U)$ does not contain a nonempty open subset of X. Then every nonempty open subset of X will intersect $X - f^{-1}(U)$ and thus $X - f^{-1}(U)$ is dense in X. Therefore $f(X - f^{-1}(U))$ is dense in $f(X)$. Let $x \varepsilon f(X - f^{-1}(U))$. Then there is a $y \varepsilon X - f^{-1}(U)$ such that $f(y) = x$. Thus $y \notin f^{-1}(U)$ and $f(y) \notin U$. Therefore $x = f(y) \varepsilon f(X) - U$. Hence $f(X - f^{-1}(U)) \subset f(X) - U$. Thus $U \cap f(X)$ is a nonempty open subset of $f(X)$ which contains no point of $f(X - f^{-1}(U))$ which is impossible since $f(X - f^{-1}(U))$ is dense in $f(X)$. Therefore $f^{-1}(U)$ contains a nonempty open subset of X and f is somewhat continuous.

<u>THEOREM 3.4</u>. If $f : (X, S) \to (Y, T)$ is somewhat continuous and A is a dense subset of X and S_A is the induced topology for A, then $f|_A : (A, S_A) \to (Y, T)$ is somewhat continuous.

PROOF: Let B be a dense subset of (A, S_A). Since A is dense in X, B is a dense subset of X. Since f is somewhat continuous by Theorem 3.3 (a) and (c) $f(B)$ is dense in $f(X)$. But $f|_A(B) = f(B) \subset f(A) \subset f(X)$. Therefore $f|_A(B)$ is a dense subset of $f(A)$ and by Theorem 3.3 (a) and (c) $f|_A$ is somewhat continuous.

REMARK 3.3. For a subset to be either open or closed is not sufficient to insure that the restriction be somewhat continuous as the following example shows.

EXAMPLE 3.3. Let $X = \{a, b, c, d\}$, $S = \{\emptyset, X, \{a, b, c\}, \{d\}\}$ and $T = \{\emptyset, X, \{c, d\}\}$. Let $A = \{a, b, c\}$. Let $f : (X, S) \to (Y, T)$ be defined by $f(x) = x$ for all $x \in X$. Then f is somewhat continuous but $f|_A$ is not somewhat continuous.

PROOF: A is both an open and closed subset of (X, S) since A and $\{d\}$ are members of S. f is somewhat continuous since $f^{-1}(X) = X$ and $X \subset X$ and $f^{-1}(\{c, d\}) = \{c, d\}$ and $\{d\} \subset \{c, d\}$. Now, $S_A = \{\emptyset, A\}$. $f|_A : (A, S_A) \to (Y, T)$ is not somewhat continuous, because $(f|_A)^{-1}(\{c, d\}) = \{c, d\}$ and $A = \{a, b, c\}$ is the only nonempty set of S_A.

THEOREM 3.5. If (X, S) and (Y, T) are spaces and $X = A \cup B$ where A and B are open subsets of X and $f : (X, S) \to (Y, T)$ is a function such that $f|_A$ and $f|_B$ are somewhat continuous then f is somewhat continuous.

PROOF: Let $U \in T$ such that $f^{-1}(U) \neq \emptyset$. Then either $f^{-1}(U) \cap A \neq \emptyset$ or $f^{-1}(U) \cap B \neq \emptyset$. Say $f^{-1}(U) \cap A \neq \emptyset$. Then $(f|_A)^{-1}(U) \neq \emptyset$ and there is an element V of S_A such that $V \neq \emptyset$ and $V \subset (f|_A)^{-1}(U)$. Then $V \in S$ and $V \subset f^{-1}(U)$.

REMARK 3.4. In the previous theorem (Theorem 3.5) it is not enough to assume that either $A \cap B$ is closed or that $A \cap B$ is open as the next example shows.

<u>EXAMPLE 3.4</u>. Let $X = \{a, b, c, d\}$, $S = \{\emptyset, X, \{a\}, \{b, c, d\}\}$, and $T = \{\emptyset, X, \{b, c\}\}$. Let $A = \{a, b, c\}$ and $B = \{a, d\}$. The function $f : (X, S) \to (X, T)$ defined by $f(x) = x$ for all $x \in X$ has the property $f|_A$ and $f|_B$ are somewhat continuous but f is not somewhat continuous.

PROOF: Note that $A \cap B = \{a\}$ is both an open and closed subset of (X, S). $S_A = \{\emptyset, A, \{a\}, \{b, c\}\}$. $f|_A : (A, S_A) \to (X, T)$ is somewhat contiuuous, since $(f|_A)^{-1}(X) = A$ and $(f|_A)^{-1}(\{b, c\}) = \{b, c\}$.

$S_B = \{\emptyset, B, \{a\}, \{d\}\}$. $f|_B : (B, S_B) \to (X, T)$ is somewhat continuous since $(f|_B)^{-1}(X) = B$ and $(f|_B)^{-1}(\{b, c\}) = \emptyset$. $f : (X, S) \to (X, T)$ is not somewhat continuous since there is no nonempty set belonging to S that is a subset of $f^{-1}(\{b, c\}) = \{b, c\}$.

<u>THEOREM 3.6</u>. If (X, S) and (Y, T) are topological spaces and A is an open subset of X and $f : (A, S_A) \to (Y, T)$ is a somewhat continuous function such that $f(A)$ is dense in Y, then any extension F of f mapping (X, S) into (Y, T) is somewhat continuous.

PROOF: Let F be an extension of f. Let $U \in T$ such that $F^{-1}(U) \neq \emptyset$. Then $U \neq \emptyset$ and since $f(A)$ is dense in Y, $U \cap f(A) \neq \emptyset$. Thus $f^{-1}(U) \neq \emptyset$. Therefore there is a $V \in S_A$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$. But since A is open, $V \in S$ and since F is an extension of f, $V \subset f^{-1}(U) \subset F^{-1}(U)$. Hence F is somewhat continuous.

REMARK 3.5. The next two examples show that both A being open in X and $f(A)$ being dense in Y are necessary in Theorem 3.6.

EXAMPLE 3.5. Let $X = \{a, b\}$, $S = \{\emptyset, X, \{a\}\}$ and $T = \{\emptyset, X, \{b\}\}$. Let $A = \{a\}$. Define $f: (A, S_A) \to (X, T)$ by $f(a) = a$. Then f is somewhat continuous and A is an open subset of X. The extension F of f defined by $F(a) = a$ and $F(b) = b$ is a function from (X, S) onto (X, T) which is not somewhat continuous.

PROOF: Note that $S_A = \{\emptyset, A\}$. $f : (A, S_A) \to (X, T)$ is somewhat continuous since $f^{-1}(X) = A \in S_A$ and $f^{-1}(\{b\}) = \emptyset$. Now consider $F : (X, S) \xrightarrow{\text{onto}} (X, T)$. $F^{-1}(\{b\}) = \{b\}$ which does not contain a nonempty open set in S. Therefore F is not somewhat continuous.

EXAMPLE 3.6. Let $X = \{a, b\}$, $S = \{\emptyset, X\}$, and $T = \{\emptyset, X, \{a\}\}$. Let $A = \{a\}$. Then A is not open in S. Define $f : (A, S_A) \to (X, T)$ by $f(a) = a$. Then $f(A)$ is dense in (X, T) and f is somewhat continuous. The extension F of f defined by $F(a) = a$ and $F(b) = b$ is a function from (X, S) onto (X, T) which is not somewhat continuous.

PROOF: $f(A) = \{a\}$ and $\overline{\{a\}} = \{a, b\} = X$. Hence $f(A)$ is dense in (X, T). $S_A = \{\emptyset, A\}$. So $f^{-1}(X) = A \in S_A$ and $f^{-1}(\{a\}) = A \in S_A$. Hence f is somewhat continuous. To see that F is not somewhat continuous, note that $F^{-1}(\{a\}) = \{a\}$ which does not contain a nonempty open set of S.

SUMMARY

In this thesis, the author examined definitions of several classes of functions weaker than continuous functions. The primary goal was to determine if these functions had elementary properties which closely parallel to the elementary properties of continuous functions, that is, compositions, restrictions and extensions.

Stallings' Almost Continuous Mapping was defined in Chapter I. From this definition, it was observed that for $gof : X \to Z$ to be almost continuous, $f : X \to Y$ had to be almost continuous and $g : Y \to Z$ had to be continuous. Also, for $gof : X \to Z$ to be almost continuous, a continuous function f had to map a compact Hausdorff space X into a Hausdorff space Y and an almost continuous function g had to map a Hausdorff space Y into a topological space Z. The theorem, $f : X \to Y$ is almost continuous and C is a closed subset of X, then $f|_C : C \to Y$ is almost continuous was proved.

In Chapter II, Singal and Singal's Almost-Continuous Mapping was defined. The main theorem was if $f : X \to Y$ the following statements are equivalent:

(a) f is almost continuous.

(b) Inverse image of every regularly-open subset of Y is an open subset of X.

(c) Inverse image of every regularly-closed subset of Y is a closed subset of X.

(d) For each point x of X and for each regularly-open neighborhood M of $f(x)$, there is a neighborhood N of x such that $f(N) \subset M$.

(e) $f^{-1}(A) \subset \text{Int}\,[f^{-1}(\text{Int}\,(\bar{A}))]$ for every open subset A of Y.

(f) $[\overline{f^{-1}(\overline{\text{Int}\,(B)})}] \subset f^{-1}(B)$ for every closed subset B of Y.

(g) For any point $x \varepsilon X$ and for any net $\{x_\lambda\}_{\lambda\, \varepsilon\, D}$ which converges to x, the net $\{f(x_\lambda)\}_{\lambda\, \varepsilon\, D}$ is eventually in each regularly-open set containing $f(x)$.

For $g \circ f : X \to Z$ to be almost-continuous, f had to be an open continuous mapping of X onto Y and g had to be an almost-continuous mapping of Y into Z; or f had to be an almost-continuous mapping of X into Y and g map Y into Z had to be a homeomorphism.

The composition of two almost-continuous functions might fail to be almost-continuous. However, every restriction of an almost-continuous mapping was almost-continuous.

If $f : X \to Y$, then for f to be almost-continuous, the following conditions had to exist:

(a) x had to be a point of X, and there must exist an open set N containing x such that the restriction of f to N be almost continuous at x.

(b) $X = X_1 \cup X_2$ where X_1 and X_2 had to be closed and $f|_{X_1}$ and $f|_{X_2}$ must be almost-continuous.

(c) $X = X_1 \cup X_2$ where $f|_{X_1}$ and $f|_{X_2}$ must both be almost-continuous at a point x that belong to $X_1 \cap X_2$.

Weakly-continuous mapping and Θ-continuous mapping were defined. It was noted that every almost-continuous mapping was weakly-continuous, but a weakly-continuous mapping might fail to be almost continuous. However, if $f : X \to Y$ was a weakly-continuous open mapping, then f was almost continuous. It was determined that every Θ-continuous mapping was weakly-continuous, but a Θ-continuous mapping might fail to be almost-continuous.

Somewhat Continuous Function was defined in Chapter III. It was observed that for $gof : X \to Z$ to be somewhat continuous, the following conditions had to exist:

(a) $f : X \xrightarrow{\text{onto}} Y$ had to be somewhat continuous and $g : Y \to Z$ had to be somewhat continuous.

(b) $f : X \to Y$ had to be somewhat continuous and $g : Y \to Z$ had to be continuous. If $f : X \to Y$ was continuous and $g : Y \to Z$ was somewhat continuous, then $gof : X \to Z$ might not be somewhat continuous.

The basic theorem was if $f : X \to Y$ is a function, then the following statements are equivalent:

(a) f is somewhat continuous.

(b) If C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper closed subset D of X such that $D \supset f^{-1}(C)$.

(c) If M is a dense subset of X, then $f(M)$ is a dense subset of $f(X)$.

The theorem, if $f : (X, S) \to (Y, T)$ is somewhat continuous and A is a dense subset of X and S_A is the induced topology for A, then $f|_A : (A, S_A) \to (Y, T)$ is somewhat continuous was proved. However, it was noted that for a subset to be either open or closed was not sufficient to insure that the restriction be somewhat continuous.

It was observed from the theorem, if $X = A \cup B$ where A and B are open subsets of X and $f : (X, S) \to (Y, T)$ is a function such that $f|_A$ and $f|_B$ are somewhat continuous, then f is somewhat continuous, it was not enough to assume that either $A \cap B$ is closed or $A \cap B$ is open.

The existence of A being open in X and $f(A)$ being dense in Y were necessary to prove the theorem, if (X, S) and (Y, T) are topological spaces and A is an open subset of X and $f : (A, S_A) \to (Y, T)$ is a somewhat continuous function such that $f(A)$ is dense in Y, then any extension F of f mapping (X, S) into (Y, T) is somewhat continuous.

It was observed that every continuous function is almost continuous, weakly continuous, and somewhat continuous, but the converse might not be true.

As a problem of further study, one might investigate whether every Almost-Continuous-Mapping is Θ-continuous using the definitions introduced by Singal [2].

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